Student Workbook

Introduction to Derivatives

Name: ______________________________ Date: ________________ Class: _________

ESSENTIAL QUESTION

How can we capture the exact rate of change of a function at a single instant — and why does that idea unlock all of calculus?

Lesson Overview

The derivative of a function f at a point x is the instantaneous rate of change of f at that point. It is defined as the limit of the difference quotient (average rate of change) as the interval shrinks to zero. Understanding the derivative as a limit of secant slopes — and connecting it to tangent lines and instantaneous velocity — gives the conceptual foundation for all of differential calculus.

$$f'(x) = \lim_{h\to 0} [f(x+h) - f(x)] / h$$

Average	Average rate of change on [a,b]: $[f(b) - f(a)] / (b - a)$ — slope of the secant line.
Instant.	Instantaneous rate of change at $x = a$: $f'(a)$ — slope of the tangent line.
Quotient	Difference quotient: $[f(x+h) - f(x)] / h$ — the expression inside the limit.
Tangent	Tangent line equation: $y - f(a) = f'(a)(x - a)$.
Velocity	If $s(t)$ is position, then $s'(t)$ is instantaneous velocity.

Worked Examples

EXAMPLE 1

Find the average rate of change of $f(x) = x^2$ on the interval [1, 3].

- Average rate of change = $[f(3) - f(1)] / (3 - 1)$.
- $f(3) = 3^2 = 9$. $f(1) = 1^2 = 1$.
- $[9 - 1] / (3 - 1) = 8 / 2 = 4$. This is the slope of the secant line through (1,1) and (3,9).

Answer: Average rate of change = 4.

EXAMPLE 2

Use the limit definition to find the instantaneous rate of change $f'(2)$ for $f(x) = x^2$.

- Difference quotient: $[f(2+h) - f(2)] / h$.
- $f(2+h) = (2+h)^2 = 4 + 4h + h^2$. $f(2) = 4$.
- $[4+4h+h^2-4]/h = (4h+h^2)/h = h(4+h)/h = 4+h$.
- $\lim_{h\to 0}(4+h) = 4$.

Answer: $f'(2) = 4$.

EXAMPLE 3

Find f'(x) using the limit definition for $f(x) = 3x^2 - 2x$.

- $f(x+h) = 3(x+h)^2 - 2(x+h) = 3x^2 + 6xh + 3h^2 - 2x - 2h$.
- $f(x+h) - f(x) = 6xh + 3h^2 - 2h$.
- Divide by h: $6x + 3h - 2$. Take the limit: $\lim(h\to0)(6x+3h-2) = 6x - 2$.

Answer: $f'(x) = 6x - 2$.

EXAMPLE 4

Write the equation of the tangent line to $f(x) = x^2$ at $x = 1$.

- Slope: $f'(x) = 2x$ (from the limit definition), so $f'(1) = 2$.
- Point: $f(1) = 1^2 = 1$, so the point is (1, 1).
- Point-slope form: $y - 1 = 2(x - 1) \to y = 2x - 1$.

Answer: Tangent line: $y = 2x - 1$.

EXAMPLE 5

A ball is thrown upward with position $s(t) = -16t^2 + 64t$ (feet). Find the instantaneous velocity at $t = 1$ second.

- $s'(t) = \lim(h\to0)\ [s(t+h) - s(t)] / h$.
- $s(t+h) - s(t) = -32th - 16h^2 + 64h$. Divide by h: $-32t - 16h + 64$.
- $\lim(h\to0)$: $s'(t) = -32t + 64$. At $t = 1$: $s'(1) = -32(1)+64 = 32$ ft/s.

Answer: Instantaneous velocity at t = 1 is 32 ft/s (upward).

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 Find the average rate of change of $f(x) = x^3$ on the interval [1, 2].

Hint: Use $[f(2) - f(1)] / (2 - 1)$. Compute $f(2) = 8$ and $f(1) = 1$, then subtract and divide.

2 Use the limit definition to find f'(x) for $f(x) = 2x + 5$.

Hint: Write $[f(x+h) - f(x)] / h$. Substitute $f(x+h) = 2(x+h)+5 = 2x+2h+5$, subtract f(x), simplify, then take the limit as $h\to0$.

3 Find f'(3) for $f(x) = x^2 - 4x$ using the limit definition.

Hint: First find f'(x) using the limit definition (expand $(x+h)^2$, simplify, divide by h, take limit). Then substitute x = 3.

4 Write the equation of the tangent line to $f(x) = x^3$ at $x = 2$.

Hint: Find $f'(x) = 3x^2$ (power rule preview). Evaluate $f'(2)$ for slope and $f(2)$ for the y-coordinate. Use point-slope form.

5 A particle's position is $s(t) = t^2 - 3t$. Find the instantaneous velocity at $t = 2$.

Hint: Find $s'(t)$ using the limit definition. Expand $(t+h)^2-3(t+h)$, subtract $s(t)$, divide by h, then take the limit. Substitute $t = 2$.

Key Vocabulary

Derivative	The instantaneous rate of change of a function at a point: $f'(x) = \lim_{h\to 0} [f(x+h)-f(x)]/h$.
Difference Quotient	The expression $[f(x+h)-f(x)]/h$, which gives the average rate of change over an interval of width h.
Tangent Line	The line that touches a curve at exactly one point and has slope equal to the derivative at that point.
Instantaneous Rate of Change	The rate of change of a function at a single instant, equal to the derivative $f'(a)$ at that point.
Average Rate of Change	The slope of the secant line between two points: $[f(b)-f(a)]/(b-a)$.

Quick Check Quiz

Circle the letter of the correct answer for each question.

1 **What is the average rate of change of $f(x) = x^2$ on [2, 5]?** Multiple Choice

- **A** 5
- **B** 7
- **C** 9
- **D** 14

2 **Which expression correctly defines the derivative f'(x)?** Multiple Choice

- **A** $\lim(h\to 0)\ [f(x) - f(h)] / h$
- **B** $\lim(h\to 0)\ [f(x+h) - f(x)] / h$
- **C** $[f(x+1) - f(x)] / 1$
- **D** $\lim(x\to 0)\ [f(x+h) - f(x)] / h$

3 **Using the limit definition, what is f'(x) for f(x) = 5x?** Multiple Choice

- **A** 5x
- **B** 5
- **C** 0
- **D** 5x + 5

4 **The slope of the tangent line to $f(x) = x^2$ at $x = 4$ is:** Multiple Choice

A 4
B 8
C 16
D 2

5 **If $s(t) = t^2 + 2t$ represents position, what is the instantaneous velocity at $t = 3$?** Multiple Choice

A 6
B 8
C 11
D 15

Common Mistakes to Avoid

✗ Confusing average rate of change with instantaneous rate of change.	✓ Average rate of change uses two points: [f(b)−f(a)]/(b−a). Instantaneous rate of change is the limit as the interval shrinks to zero: f'(a) = lim(h→0)[f(a+h)−f(a)]/h.
✗ Forgetting to subtract f(x) in the numerator of the difference quotient.	✓ The difference quotient is [f(x+h) − f(x)] / h. Both f(x+h) AND f(x) must appear in the numerator.
✗ Canceling h before simplifying the numerator fully.	✓ Expand and combine all terms in the numerator first so that every remaining term has a factor of h, then cancel h.
✗ Using the wrong point when writing the tangent line equation.	✓ The tangent line at x = a passes through (a, f(a)). Compute f(a) — do not use f'(a) as the y-coordinate.

Math Tips

★ The difference quotient $[f(x+h)-f(x)]/h$ is just the slope formula (rise/run) with run = h.

★ After expanding $f(x+h)$, every term without h should cancel with $f(x)$ in the numerator.

★ The power rule $f'(x^n) = nx^{n-1}$ is a shortcut derived from the limit definition — it always gives the same answer.

★ A positive derivative means the function is increasing; a negative derivative means it is decreasing.

★ Velocity is the derivative of position; acceleration is the derivative of velocity.